RELATIVELY SPEAKING

BY FREDERICK V. HUNT, '25.

EINSTEIN'S theories of relativity were developed in two parts. The first, published in 1905, was called the Special Theory of Relativity, followed ten years later, in 1915, by the General Theory of Relativity. The mathematical complexities involved in this latter generalization were the facts that led Dr. Einstein to distinguish the twelve men who could understand it. This exaggeration greatly confused the situation. While an understanding of the General Theory is impossible without an unusually thorough mathematical training, a fundamental idea of the assumptions and the easier conclusions of the Special Theory are well within the mental grasp of the moderately scientific mind.

The Special Theory of Relativity is based on two postulates which it will be well to state at the outset:

Postulate I: "Every law of nature which holds good with respect to a co-ordinate system K must hold good for any other system K′, provided that K and K′ are in a uniform movement of translation."

Postulate II: "Light in a vacuum has a definite and constant velocity, independent of the velocity of its source."

The implications involved in the first of these are many. It is to say, all motion is relative; that we may not definitely draw any conclusions about a moving body unless we too can move with it, that is, reduce the relative motion to zero (except as we make allowance for this in our observations, as will presently be done). Take, for instance, a train moving on a long straight track. To an observer in the train his position represents a system of co-ordinates, say K, in motion with regard to the track. If now another train on a track beside the first is speeding along at the same velocity, so that they remain side by side, to an observer on that train it will constitute a second set of co-ordinates, which we can call K′, in motion relative to the track but in the same state of uniform translation as the first. This postulate then assumes that any law of nature or observed behavior which holds good in one moving train will also hold good with regard to the other moving train beside it. This is a perfectly natural conclusion and a statement that we should never naturally challenge. However, notice that it does *not* imply that laws which hold good for one train K will also hold good for the other train K′ *if it should stop*, or indeed for any other system of co-ordinates which is not in the same state of uniform translation. This is a very common error and one which the Special Theory corrects. Because when on the second train K′, we measured the length of train K while both were moving, we assume that a person standing by the track (co-ordinate system not in motion) would find the same length by measurement. The relativity theory in correcting this obvious logical error seems to contradict our common sense idea of things. I propose to demonstrate which is the more logical.

The second principle on which the Special Theory rests is the constancy of the velocity of light in vacuum. Physicists owe their confidence in this proposition to both the Maxwell-Lorentz theory of electro-dynamics and the marvelous experimental work of the American physicists, Michelson and Morley. Their empirical verifications of this postulate are most convincing. Tests were made with sources of light in all types of motion. They used a star in its orbit, both when approaching the earth and when receding from it. They used sources on the laboratory table which were in rapid translation both rotationally and in its path about the sun. But always the same velocity of light was determined. Their instruments were delicate enough to have detected a difference of velocity much smaller than that attained between the approaching and receding speeds of the planet sources used. It seemed impossible to give light a swifter "send off" by speeding up the motion of the source. Take for example the case of an approaching train. A baseball thrown from the cowcatcher would have the compounded velocities of throwing and of the train. But the velocity of the beam projected from the headlight was found to be constant! Likewise when the train is withdrawing a baseball thrown from the back platform would have the compounded velocities of the train (negative this time) and of throwing and consequently would travel slowly, while the velocity of the light from the disappearing tail light remains serenely constant.

These two principles have received such strong experimental confirmation as to be almost unquestionable, yet they do not, at first, appear to be logically compatible. Einstein says, in the London "Times," 1919: "The Special Relativity Theory achieved their logical reconciliation by making a change in kinematics, that is to say, in the doctrine of the physical laws of space and time."

The problem suggested, of two trains in uniform motion, is very interesting. However, when we said that the length of the moving train could not be ascertained by a person (system of co-ordinates) at rest, we have suggested a much more absorbing problem. Let us solve it.

Consider Fig. 1, our illustration of a railway train R, moving with velocity V, for instance, at 60 miles per hour relative to the track B.

We will denote distance relative to the train—that is measured in the train—by x', and time in the train by t'. The distances measured along the track may be denoted by x and the time on the track by t. For simplicity we may count distance and time in the train and on the track from the same zero value—that is, assume $x=0$, $x'=0$, $t=0$, $t'=0$. This can make no essential difference and will eliminate unnecessary constant terms from the equations of transformation from the train to track and track to train. x' and t', the co-ordinates with regard to the train are thus moving at velocity v relative to the co-ordinates x and t, the co-ordinates

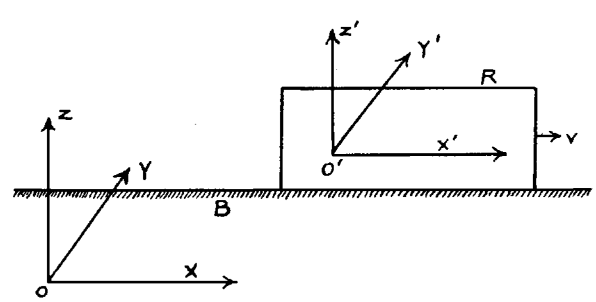

FIG. 1

with regard to the track, and by the conventional Newtonian mechanics we should have $t'=t$. That is, time would be the same on the track as in the train and distance along the track would be

$$x'=x+vt'.$$

That is, in the conventional Newtonian mechanics, distance along the track, x, increases in the time t by the velocity v. These equations obviously do not apply, however, as they would give different velocities of light relative to the track and relative to the train, as in the case of a thrown ball. This of course is no longer permissible, as we have not only postulated but experimentally proven the constancy of the velocity of light. We must therefore start with the most general relations between x and x', t and t', and pursue them to their logical consequences in the light of our basic postulates.

Three conditions must be met:

1. The relative velocity of the train co-ordinates x' and t' with regard to the track co-ordinates x and t is v.
2. The relative motion of the train with regard to the track is the same as the relative motion of the track with regard to the train,—that is the same equations relate x and t to x' and t' as relate x' and t' to x and t. (This is a logical and mental necessity and by a very rigorous connotation of the term relative velocity would not need to be stated. It is well however to be explicit.)
3. The velocity of light, c, on the track in the x, t co-ordinates is the same as the velocity of light in the x', t' co-ordinates.

In the most general expression, the train co-ordinates x', t' are related to the track co-ordinates x, t by the co-ordinate transformation equations:

$$x'=ax-bt.$$
$$t'=pt-qx. \qquad \text{(a)}$$

(The relation must be linear as it is univalent,—that is, one point of the train can correspond to one point of the track only, and inversely.)

1. When $x'=O$: $(ax-bt)=O$: but since x', t' have to x, t the velocity v, $x/t=v$ and it follows that

$$b/a=v$$
$$b=av \qquad \text{(b)}$$

Thus

$$x'=ax-avt\ .$$
$$t'=pt-qx\ . \qquad \text{(c)}$$

4. From the condition of relativity it follows from equations (a)

$$x'=ax'+avt'\ .$$
$$t=pt'+kx\ . \qquad \text{(d)}$$

The reversal of sign is obvious since the motion of the train, relative to the track is clearly opposite to that of the track, relative to the train.

Substituting (c) into (d) gives:

$$x(a^2-avq-1)+avt(p-a)=O$$
$$t(p^2-avq-1)+qx(p-a)=O$$

As these must be identities and hold for *any* value of x or t, the coefficients of x and t must individually vanish: that is

$$(p-a)=O$$
$$p=a$$
$$(a^2-avq-1)=O$$
$$q=\frac{a^2-1}{av} \qquad \text{(e)}$$

Substituting these in (c) gives

$$x'=ax-avt$$
$$t'=at-\frac{(a^2-1)}{av}\,X \qquad \text{(f)}$$

3. From the constancy of the velocity of light this relation must hold:

$$x=ct$$
$$x'=ct' \qquad \text{(g)}$$

Substitute (g) into (f); divide, cancel and simplify as follows:

$$ct'=a(ct-vt)$$
$$t'=at-\frac{(a^2-1)}{av}\,ct$$

Divide

$$c=\frac{ac-av}{a-\dfrac{(a^2-1)}{av}\,c}$$

Transpose, cancel, and simplify into

$$a^2(c^2-v^2)=c^2$$
$$a=\sqrt{\frac{c^2}{c^2-v^2}}$$
$$=\frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

Substituting in (d) and (c) the constants are all determined and the four transformation equations become:

$$x=\frac{x'+vt'}{\sqrt{1-\dfrac{v^2}{c^2}}}\ :\quad x'=\frac{x-vt}{\sqrt{1-\dfrac{v^2}{c^2}}} \qquad (1)$$

$$t=\frac{t'+\dfrac{v}{c^2}x'}{\sqrt{1-\dfrac{v^2}{c^2}}}\ :\quad t'=\frac{t-\dfrac{v}{c^2}x}{\sqrt{1-\dfrac{v^2}{c^2}}} \qquad (2)$$

There are (at least) three rather startling conclusions to be drawn from a consideration of these equations.

1. Simultaneity is a relative thing and two events which occur simultaneously to one observer, *A*, may happen in a certain sequence to another observer, *B*, and in a reversed sequence to a third observer, *C*. Take for examples two events occurring in the train at the same time, t' that is simultaneously, but at two different points x'_1 and x'_2 of the train. They will not be simultaneous as viewed from the track, but they will occur at two different times as given by the equations:

$$t_1=\frac{t'+\dfrac{v}{c^2}x_1'}{\sqrt{1-\dfrac{v^2}{c^2}}} \quad \text{and} \quad t_2=\frac{t'+\dfrac{v}{c^2}x_2'}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

That is, if the engineer, x_2, and a passenger on the rear platform, x_1, clap their hands at the same time, t', an observer on the track sees the passenger at the rear clap his hands first. (If the engineer is x_2 then $x_2 > x_1$ and t_2 will exceed t_1, and hence occur later.) Suppose, however, that the man on the rear platform claps his hands, not simultaneously with the engineer but a trifle later, such that the time difference is less than that between t_2 and t_1. Then the observer on the track would see the man on the rear platform (event x_1) clap his hands first and the engineer (event x_2) later, while an observer on the train would see the engineer clap his hands first and the man at the rear later.

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In other words, simultaneousness in time or occurrence is only relative and two events that appear simultaneous to one observer may not be simultaneous to another observer because of a different relative motion.

2. The distance between two points p_1 and p_2 of the train in train co-ordinates, that is, as seen from the train is—

$$L' = x_2' - x_1' \quad :$$

in track co-ordinates, that is, as seen from the track, the same distance is—

$$L = x_2 - x_1 \quad .$$

However, by equations (1),

$$x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or

$$L = L' \sqrt{1 - \frac{v^2}{c^2}}$$

This is to say, a length L' appears from the track to be shorter by the factor—

$$\sqrt{1 - \frac{v^2}{c^2}}$$

(the more, the faster the speed).

If the train were to move with the velocity of light, $v=c$, the length L' in the train would appear from the track as $L=O$, that is, it would vanish. If the train should reach a speed greater than that of light the length L would become imaginary. From this it is deduced that no velocity can exist greater than that of light!

This conclusion is the foundation of many popular sayings concerning the variation in the length of a yardstick. And indeed, a stationary observer (if any)

would agree that a yardstick appears shorter when placed East and West than when placed North and South, due to the rotational motion of the earth. We can, however, not detect any such phenomenon because any standard or device by which we attempt to compare the varying yardstick will itself suffer equal distortion.

3. Time is relative. The time difference between two events occurring at a point P in the train, by the time as noted by an observer in the train, that is in train co-ordinates is—

$$T' = t_2' - t_1' \quad ;$$

but seen from the track, that is for an observer on the track watching the clock in the train while standing in the track co-ordinates, the time difference between the two events becomes

$$T = t_2 - t_1 \ .$$

However, by equations (2),

$$t_2' - t_1' = \frac{t_2 - t_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or

$$T' = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}}$$

To the observer from the track, the time on the train, that is, the duration between two events on the train, has been slowed down or the duration lengthened by the factor—

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We have been long accustomed to think of the ceaseless, steady march of time. Indeed, it has been one of our most fundamental concepts,—"The unrelenting march of Moments, each one here an instant, then is gone, forever." But it now seems that time itself is a relative thing and that the poet's insight was not so sound as seemed at first.

This explains the statement, often alluded to, concerning the fact that a man moving with the speed of light never grows old. For if $v=c$, that is, if the velocity of the train were equal to that of light, the time in the train would be slowed down by the factor

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \ .$$

$$= \frac{1}{0} = \text{infinity},$$

that is, the duration of time between any two events under such conditions would be infinite,—"for the clock has stopped."

If it were possible to project a rocket off into space at a speed of 1/20,000th less than that of light, the passenger there would have some very interesting experiences. If in, say a year (measured in the rocket) his carrier were caught in the field of some distant star and like a comet swung back upon its path toward the Earth, on alighting he would find himself only two years older, while the Earth, traveling at no such violent speed, would have aged some 200 years. A very interesting way to study history but not exactly reassuring!

Incidentally it is consoling to remark that it requires the expenditure of an infinite amount of energy to give even the smallest particle of matter the velocity of light. But that is another story.

An interesting question arises out of a consideration of our conclusion regarding the non-existence of velocities greater than that of light. Particularly in the field of ionic velocities, which often closely approach that of light, we might find the situation where two ions, each moving with 0.9 of the velocity of light, are moving in opposite directions. By our conventional Newtonian concepts we would imagine them compounding to give a relative velocity of almost twice that of light. However a study of our fundamental equations* shows that here too, the theory is consistent. The velocity compounded of two separate velocities 90% that of light is a resultant velocity 99.45% of the velocity of light. From this it follows that as long as the respective velocities are less than c, no matter how closely they may approach it, their sum too will be less than c.

If, however, one of the velocities equals the velocity of light, c, then substituting it in the equation yields a resultant also of c. That is, adding or subtracting any velocity, v_1, to or from the velocity of light, c, still gives the same resultant. This explains, as nearly as a theory can explain its own postulates, why in the previous case, if a train moves at the velocity v and the light along the track at the velocity c, the velocity of the light relative to the train will be the combination of c and v, which is still c.

The velocity of light, then, has the characteristics of the mathematical conception of infinity: that is, any velocity, whether less than or equal to c may be added to it or subtracted from it without the velocity of light itself changing.

This, in itself, logically demonstrates the impossibility of having any velocity greater than that of light since whatever velocity may be added to it, it remains unchanged at c.

The more fundamental propositions expounded here have passed beyond the stage of doubtful conjecture into the field of probable actuality. Of the three testing points Einstein advanced, he has been borne up in two of them; notably, in explaining the shifting of the perihelion of Mercury, a thing that Newton's law of gravity did not adequately do; and in explaining the deviation of the light from stars, caused by the gravitational field of the Sun. The third point,—the shifting of the spectrum of certain stars toward the red, lies yet in the balance. Sir Oliver Lodge says, "If Einstein's third prediction is verified, his theory will dominate all higher physics and the next generation of physicists will have a terrible time of it."

These suggestions of terrifying mathematics, and the fourth dimension, present an irresistible appeal to the philosophical scientist and offer a splendid challenge to the limitations of our feeble human concepts.

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* Note.—From (1) and (2): $v = \frac{x}{t} = \frac{v_1 + v_2}{1 + \frac{v_1 \ v_2}{c}}$